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Description of the "basket-of-zeros" discounting method and the derivation of present value factors from the yield curve

A few definitions

The "***basket-of-zeros***" discounting method defines the present value of a series of payments as the value today of a collection of zero-coupon bonds that, at maturity, exactly match the cash flow observations. The methods used to derive present value factors from the yield curve and use them to determine present value are described below.

Coupon or "bond-equivalent" rate. Treasury notes and bonds pay interest every six months. This payment, called the "coupon" for historical reasons, is computed using the "coupon rate" divided by 2. Thus, a Treasury bond in the amount of $1,000 with a coupon rate of 6 percent would make a coupon payment of $30 every six-months. The coupon rate does not take into account that the interest earnings might be reinvested.

Effective annual yield. This is the yield the bond holder would receive if interest earnings were reinvested at the coupon rate. For example, using the example above and assuming that the $30 coupon payment can be reinvested at 6 percent, the annual yield would be 6.09 percent. The effective annual yield is computed as $[\ 1 + (\text{ coupon rate } / \ 2)\ ]^{2} - 1$.

Zero-coupon bond. If a bond has a coupon rate of zero percent (pays no explicit interest) and the face value is paid at maturity, it is called a zero-coupon bond. Ordinarily, Treasury limits the issuance of such securities to periods of less than one year. However, they are available indirectly available through the "STRIPS" (Separate Trading of Registered Interest and Principal Securities) program, by which holders of eligible securities may trade coupon or principal payments separately in the secondary market or combine them.

Spot rates. The spot rate for a particular maturity is the bond-equivalent rate on a zero coupon bond of the same remaining maturity. Spot rates can be obtained by direct observation of zero-coupon bonds in the secondary market. Also, they can be derived from the observed yields on unstripped notes and bonds, as described below. Because the market for "STRIPS" is not as extensive as the market for Treasury securities in unstripped form, the second method is generally used to obtain spot rates. Spot rates are stated as bond-equivalents to facilitate comparison with other Treasury rates.

Bond price. The theoretical price of a bond in the secondary market is determined by calculating the present value of each coupon payment and, at maturity, the payment of principal, using a different spot rate for each payment. A payment in 6 months would be discounted using the 6-month spot rate, a payment in 12 months would be discounted using the 12-month spot rate, and so forth. If the principal payment were to be received at the end of 10 years, it would be discounted using the 10-year spot rate. The sum of the present values of the individual payments, so calculated, is the theoretical price of the bond.

Yield-to-maturity. This is the bond-equivalent rate that an investor would need to receive on all coupon payments to make the price of a bond equal to par.

Present Value factors. To determine the present value of a single payment, the payment is multiplied by the appropriate present value factor. This factor, in turn, is derived from the spot rate for the maturity of that payment. For example, the present value of a single payment of $1,000 at the end of three years would be computed by first deriving a present value factor by:

$$\text{Present value factor} = 1 / [\ 1 + (\ \text{spot rate} / 2)\ ]^{6}$$

and, multiplying the payment by that present value factor. If the spot rate at the three-year point is 8 percent, the present value would be calculated as follows:

$$\text{Present value factor} = 1 / [\ 1 + (\ 8.00 / 200\ )\ ]^{6}$$

$$= 0.79031$$

$$\text{Present value} = \text{Cash value} \bullet 0.79031$$

$$= 1000 \bullet 0.79031$$

resulting in a present value of $790.31.

Present value. For a series of payments, which may vary in amount and timing, the present value is the price of a collection of zero-coupon bonds that, at maturity, exactly match the series of payments. Present values can also be calculated using a single discount rate; however, without a clear basis for selecting a discount rate (for example, "similar maturity" is open to varying interpretation), the relationship between nominal and present values is less clear.

Forward rate. The forward rate is the reinvestment rate (generally on six-month bills) that would equalize earnings on a long-term security (usually at higher rates) with series of short-term (generally six-month) securities. For example, if the six-month rate is 6.00 percent and the one-year rate is 6.50 percent, then the forward rate in six months would be 7.00 percent (calculated from the ratio of exponentiated spot rates, or $(\ (\ 1.0325^2 / 1.03\ ) - 1\ ) \bullet 200$).

Conventions

The following conventions are used in this document:

Interest rates are consistently shown in decimal form (“0.06” rather than “6.0 percent”).

Multiplication is indicated by the “•” symbol. For example, 2 • 3 = 6.

Symbols (time periods denoted by the subscript “n” are semiannual periods unless otherwise stated):

C_i — Coupon payment, per $1,000 of principal, in period i (if the coupon rate is 5.4 percent, C_2 is $27.00)

F_i — Forward rate for the time period i

L_i — Logarithmic factor in period i where L_1 is log(1) and has a value of zero, L_2 is log(2) and has a value of 0.301, and L_n would have a value of log(n).

P_i — Present value factor for semiannual period i, computed as:

$$P_i = \prod_{k=1}^{i} [\ 1/(1 + (\ F_k / 2\ ))\ ]$$

If F_1 is 0.052 (5.2 percent), P_1 would be .9747.

Present value factors can also be calculated directly from spot rates where the present value factor in semiannual period n would be calculated from the spot rate, stated in bond-equivalent terms, for time period n (S_n) by:

$$P_n = 1 / [\ 1 + (\ S_n / 200\ )\ ]^n$$

P_{fti} — Present value factor for period i, with frequency f, and timing t. Frequency can be monthly, quarterly, semiannually, or annually. Timing can be beginning, middle, or end of the period, or throughout the period (equivalent to occurring mid-period).

X_{fti} Cash flow observation in period i, with frequency f, and timing t. Frequency and timing as defined above.

Y_i Yield-to-maturity rate for period i

Published yield curve data

The starting point for the derivation of spot rates and discount factors is the published points on the Treasury yield curve. This yield curve shows yield-to-maturity rates, derived from the prices at which securities are traded in the secondary market, arrayed against the remaining maturity of the securities. The Department of the Treasury and the Federal Reserve routinely publish quotations for selected points on the yield curve. The publications are cited below.

As of this writing, the published yield curve points are:

3 months
6 months
12 months
2 years
3 years
5 years
7 years
10 years
30 years

The sections below describe how these published yield-to-maturity observations are translated into a collection of present value factors that can be used in the credit subsidy calculator.

Discounting requirements

The credit subsidy calculator needs to discount cash flow estimates that:

May be stated in annual, semiannual, quarterly, or monthly intervals;

May, within such periods, have activity concentrated at the beginning, middle, or end of those periods; and,

May extend for up to 100 years (a little over twice the term of the Federal credit program with the longest term (48 years)). See the *Federal Credit Supplement* to the *Budget of the United States Government, Fiscal Year 2000*, tables 3 and 4, for details.

Overview of the derivation of present value factors

The conversion of published yield curve points to twice-monthly forward rates and present value factors takes place in several steps.

> For the interval from 6 months to 30 years, forward rates and their associated present value factors are derived at semiannual points from the published yield curve;
>
> Semiannual spot rates are derived from semiannual present value factors;
>
> In the same interval, twice-monthly spot rates are interpolated from the semiannual spot rates and twice-monthly present value factors are derived;
>
> For the interval from 30 years to 100 years, forward rates are held constant (at the value for the forward rate at 30 years) and present value factors and spot rates are derived; and
>
> For the interval from 0 months to 6 months, spot rates are interpolated (3 months to 6 months) and estimated (0 months to 3 months).

The methods used in each of these steps are discussed below.

How forward rates and present value factors are derived from the yield-to-maturity curve in the interval from 6 months to 30 years

The six-month forward rate and six-month spot rates are equal to the observed six-month yield rate.

The one-year present value factor and forward rate are derived from the values of P_1 (present value factor, first semiannual period) and C_2 (coupon rate, second semiannual period), which are known from the published data. The value of P_2 can be determined from the following condition.

$$1000 = (1000 \cdot P_2) + [C_2 \cdot (P_1 + P_2)]$$

which states that the present value of the stream of payments produced by the security is par. Because the observed points are based on trading in the secondary market, the condition is reasonable.

Rearranging,

$$P_2 = [1000 - (C_2 \cdot P_1)] / (1000 + C_2)$$

Given the rates above:

$$P_2 = [\ 1000 – (\ 27 \bullet .9747\ )\ ]\ /\ (\ 1000 + 27\ )$$

or 0.9481

The present value factor, P_2, can be converted to the forward rate, F_2, as follows:

$$F_2 = [\ (\ P_1 / P_2\ ) - 1\ ] \bullet 2$$

or 0.05611 or 5.611 percent.

Calculation of the present value factor for 1.5 years (P_3) is more complicated because there is no direct observation for the third period and P_3 must be found by interpolation.

Interpolation is simplified by the observation that successive forward rates generally follow a logarithmic pattern (a proportional increase in term is associated with an equal proportional increase in the rate)[1].

The first point to be interpolated is the 18-month (F_3) point, which is derived from the to-be-determined value of F_4 as follows:

$$F_3 = F_2 + [\ (\ F_4 – F_2\ ) \bullet (\ L_3 – L_2\ )\ /\ (\ L_4 – L_2\ )\ ]$$

Though the value of F_4 is unknown, it can be determined from the price of a two-year security with a stated yield-to-maturity rate that should result in a bond price of par. Specifically:

$$1000 = (\ C_4 \bullet P_1\ ) + (C_4 \bullet P_2\ ) + (C_4 \bullet P_3\ ) + [\ (\ 1000 + C_4) \bullet P_4\ ]$$

Where:

$$P_3 = [\ 1\ /\ (\ 1 + F_3\ /\ 2\ )] \bullet P_2$$
$$P_4 = [\ 1\ /\ (\ 1 + F_4\ /\ 2\ )] \bullet P_3$$

There is no direct way to solve these equations. The solution must be determined by a trial-and-error method in which successive values are tried for the two-year forward rate, F_4. For each test value, a value of F_3 and present value factors, P_3 and P_4, are computed, and the price of the two-year security is found. If the price is below par, the value of F_4 is

[1] Obviously, there have been times when the forward rate curves were inverted (short-term rates above long-term rates) or dome-shaped (medium-term rates were higher than either short- or long-term rates). However, these exceptions have been infrequent. In addition, the interpolation methods, described below, perform reasonably well for the intervals between published yield curve points even when the overall curve has an unusual shape.

lowered; if above par, it is raised until a value for F_4 is found that makes the price exactly equal to par. This process is repeated until a solution is found[2].

This method is repeated for the interval between the next pair of published yield curve points (2 years and 3 years), then the next (3 years to 5 years), and so forth, until the semiannual forward rates at all semiannual points, from 6 months to 30 years, have been found.

How twice-monthly present value factors are calculated

Once a full set of forward rates has been calculated (semiannual from 6 months to 30 years), the spot rates are derived from the present value factors, as follows:

$$S_n = [\, 2 \bullet (1/P_n)^{(1/n)} \,] - 2$$

Twice-monthly spot rates are then interpolated from the semiannual spot rates using the logarithmic method described above (z is a fraction of a period between sequential periods x and y):

$$S_z = S_x + [\, (S_y - S_x) \bullet (L_z - L_x) / (L_y - L_x) \,]$$

The twice-monthly spot rates are then converted back to present values:

$$P_z = [\, 1 / (1 + S_z / 2)^z \,]$$

How present value factors are projected from 30 years to 100 years

Forward rates after 30 years are held constant at the calculated 30-year forward rate. The corresponding twice-monthly present value factors are calculated as follows:

$$P_n = P_{n-1} \bullet (1 / ((1 + F_{60} / 2)^{(1 / 12)}))$$

[2] The algorithm used for this trial-and-error method is interval bisection method. The target rate is assumed to occur between a lower limit of -199.0 percent and an upper limit equal to the largest value that can be represented in an 8-byte floating point format or approximately 10^{308}. The midpoint of that interval is found and tested. If the midpoint estimate is too high, the upper half is discarded and the process repeated on the remaining interval. A solution is found when the mid-point value test results in a par-valued security or when the difference between the upper and lower limits of the remaining interval is zero or too small to be represented in 8-byte floating point format (roughly 15 decimal digits of precision). This method is described in more detail in *How the single effective rate is calculated*, which has several references on numerical methods.

How spot rates and present value factors prior to 6 months are derived

The spot rates from 3 months to 6 months are interpolated from the observed 13- and 26-week rates. Present value factors are derived from these spot rates.

Prior to 3 months, the twice-monthly spot rates are calculated through a series of linear equations. The equations were calculated through regression analysis of historical Treasury Bill quotes available in major newspapers. Twelve equations were calculated representing weeks one through 12. Six of these equations (or equations directly related to them) are used to calculate spot rates at the desired twice-monthly points.

The data for the regression estimates were drawn from a random sample of 100 daily observations (13 rates per observation). The sample frame was the quotations in the Wednesday newspapers from January 1968 to June 1998[3]. Because Treasury Bills mature on Thursday, the quotes in the Wednesday papers gave data points at exactly 8 days, 15 days, … and 85 days to maturity. When consecutive quotes (i.e., rates on securities maturing one week apart) differed by more than eight percent, the observation for that day was dropped. This relatively modest constraint eliminated 28 percent of the observations in the sample. Apart from the desire to remove observations with erratic period-to-period changes, there was no specific basis for choosing the 8 percent change rule. Obviously, a different rule would have a different effect on the data used for analysis, though, within reasonable limits, little effect on the estimated parameters.

Twelve equations were then calculated by regressing each of the weekly rates on the 13-week rate using the form $y = ax + b$. The parameters, based on the 2-, 4-, 9-, and 11-week rates, are used to calculate directly the twice-monthly spot rates. (The 1.5-month rate is calculated from an equation representing the average of the 6- and 7-month equations. An over-night rate (time zero) equation was calculated from the slope of the 1- and 2-week equations.). The twice-monthly regression equations are:

Rate[4]	Equation
Time zero	$= 1.0078 \cdot S_{.25} - 0.57439$
0.5 months	$= .96868 \cdot S_{.25} - 0.26107$
1.0 month	$= .94362 \cdot S_{.25} - 0.07258$
1.5 months	$= .97198 \cdot S_{.25} - 0.10305$
2.0 months	$= .98595 \cdot S_{.25} - 0.06110$
2.5 months	$= .99551 \cdot S_{.25} - 0.04864$

[3] In order to verify that the regression equations were not biased by the use of Wednesday observations, a separate (smaller) sample of Tuesday observations were collected and analyzed. The regression equations produced by the Tuesday data were compared to the Wednesday equations via Chow tests (1 vs. 1, 2 vs. 2, etc.). No significant differences were found between the equation coefficients.

[4] These are approximations because the effort needed to obtain quotes that are exactly 0.5 months, 1.0 months, …, and 2.5 months to maturity would exceed the value that the added precision might offer. Thus, 15 days is used as a reasonable approximation of 0.5 months; 29 days for 1 month; the average of 43 and 50 days for 1.5 months; 64 days for 2 months; and 78 days for 2.5 months.

where $S_{.25}$ is the 13-week spot rate.

At this point, present value factors are available for the beginning, middle, and end of each month, for 100 years.

How present value factors are used to compute present values

Cash flow observations may have monthly, quarterly, semiannual, or annual frequencies. Within those frequencies, activity may be concentrated at the beginning of the period, end of the period, or throughout the period (equivalent to middle of period). Thus, from this collection of twice-monthly discount factors, 12 subsets are used for computing present values:

Frequency	**Within-period timing**
Monthly	Beginning of the month Throughout the month (or mid-month) End of the month
Quarterly	Beginning of the quarter Throughout the quarter (or mid-quarter) End of the quarter
Semiannually	Beginning of the semiannual period Throughout the semiannual period (or mid-semiannual period) End of the semiannual period
Annually	Beginning of the year Throughout the year (or mid-year) End of the year

The present value of the cash flow observations is computed by summing the products of the observation for a particular frequency (f), timing (t), and period (n) by the present value factor for the same frequency, timing, and period:

$$\text{Present value} = (X_{ft1} \bullet P_{ft1}) + (X_{ft2} \bullet P_{ft2}) + \ldots + (X_{ftn} \bullet P_{ftn})$$

The present value, in this computation, is the market price of a collection of zero coupon bonds that, at maturity, exactly match the amounts and maturities of the cash flow observations; hence, the term “basket-of-zeros.”

This method is superior to using a constant rate for discounting because the present value of each payment is based on an instrument of unambiguously “similar maturity” for that payment. This calculation avoids an obvious anomaly produced by the previous method in which two loan guarantee programs with identical cash flows would result in different subsidies whenever the loans they guaranteed had differing maturity. With the basket-of-zeros method, two programs with identical cash flows would have identical subsidies.

Sources of Information on Treasury Interest Rates and References

Market yields on Treasury marketable securities are published by the Board of Governors of the Federal Reserve System and by the Department of the Treasury:

> Department of the Treasury, Financial Management Service, *Treasury Bulletin*, Government Printing Office. This quarterly publication is available from the Government Printing Office bookstores and from the web site of the Financial Management Service (www.fms.treas.gov).
>
> Board of Governors of the Federal Reserve System, Federal Reserve Statistical Release H.15, "Selected Interest Rates." This weekly release may be obtained from the FRB website (www.bog.frb.fed.us).

There are a number of references that can be consulted for more information on discounting practices. The following is one of the many available:

> Frank J. Fabozzi, ed., *The Handbook of Fixed Income Securities,* fifth edition, Irwin Professional Publishing, 1997.

In addition, numerous articles on this and closely related subjects have been published in academic and professional journals. The following list is by no means complete:

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